A master equation approach to option pricing

D. Faller [a] F. Petruccione [a,b]

[a] *Albert-Ludwigs-Universität, Fakultät für Physik, Hermann-Herder Straße 3, D–79104 Freiburg im Breisgau, Federal Republic of Germany*

[b] *Istituto Italiano per gli Studi Filosofici, Palazzo Serra di Cassano, Via Monte di Dio 14, 80132 Napoli, Italy*

Abstract

A master equation approach to the numerical solution of option pricing models is developed. The basic idea of the approach is to consider the Black–Scholes equation as the macroscopic equation of an underlying mesoscopic stochastic option price variable. The dynamics of the latter is constructed and formulated in terms of a master equation. The numerical efficiency of the approach is demonstrated by means of stochastic simulation of the mesoscopic process for both European and American options.

1 Introduction

As is well-known the seminal Black-Scholes analysis [1] which leads to the fair value of an option is based mainly upon the following assumptions. First the stock price, i.e. the underlying, follows a geometric Brownian motion. Second a hedge position is formed with a portfolio of short underlying and a long position of a number of European options. Then an arbitrage argument leads to the renowned Black–Scholes partial differential equation determining the value of the option. Of course, for simple cases, e.g. constant interest rate and volatility, explicit analytical solutions to the equation are known [2]. However, for more involved cases one has to rely upon numerical methods. Here we are not concerned with deterministic methods, e.g. finite differences, but rather with Monte Carlo methods [3].

Email addresses:
`Daniel.Faller@physik.uni-freiburg.de` (D. Faller),
`Petruccione@physik.uni-freiburg.de` (F. Petruccione).

The idea behind the canonical Monte Carlo method is to exploit the fact that the fair value of an option is given by the present value of the expected payoff at expiry under the risk neutral measure. Thus the standard Monte Carlo approach is based upon the simulation of a geometric Brownian motion for the underlying asset until expiry. Then the payoff is computed and discounted up to the current time. By averaging over different realizations of this stochastic process the current option price is estimated [4]. In spirit this approach is similar to the standard Monte Carlo random walk technique for the solution of partial differential equations with boundary value problems [5].
Monte Carlo methods offer easy to understand numerical algorithms which can be easily applied to quite complicated, e.g. path dependent or correlated multi-asset, options [6]. In addition they allow for a straightforward inclusion of stochastic terms such as the interest rate or volatility. From a numerical point of view they are especially suited to problems with many degrees of freedom and the algorithm can be easily run in parallel.
However Monte Carlo simulations of the Black-Scholes equation are usually slower than comparable finite difference solutions of the partial differential equation. Another disadvantage of the standard Monte Carlo approach is that while the application to European options is straightforward, the valuation of American options is more involved. Using a generalisation of the canonical Monte Carlo method one has to assure that the early exercise condition stating that the option price is always above the current payoff is not violated sometime in the future. If this happens and the option value is below the payoff, the option is exercised and the option value is given by the payoff function. A Monte Carlo simulation based on the dynamics of the underlying asset has to keep track of all these points in asset time space which makes the algorithm ineffective. Nevertheless more advanced Monte Carlo algorithms are available [7,8,9,10].
Since the Monte Carlo methods described above are based on a model of the underlying dynamics of asset values these approaches could be named microscopic. In contrast now we are going to follow a different strategy, which we would like to name mesoscopic, which has already been applied with success to simulations of turbulence in fluid dynamics [11,12,13], to the investigation of hydrodynamic fluctuations [14,15] as well as to magnetohydrodynamics [16]. The same approach has been shown to lead to fast Monte Carlo algorithms for the balance equations of nonequilibrium thermodynamics [17], for the simulation of chemical reactions [18] and reaction–diffusion processes [19].
The basic idea of the mesoscopic approach [20] is to regard the value of the option, say V as the expectation value of an underlying fluctuating (mesoscopic) stochastic option value, say θ. A master equation for θ is easily constructed in such a way that the expectation value of the multivariate stochastic variable θ satisfies the macroscopic Black–Scholes equation. The stochastic process defined in terms of a master equation is easily simulated and allows for a Monte Carlo algorithm for the direct simulation of the Black–Scholes equation.
This paper is structured as follows. In section 2 the mesoscopic master equation

approach is motivated. The dynamics of the underlying asset is modelled as a piecewise deterministic process (PDP). The derivation of the fair option price for a mesoscopic stochastic option value, which parallels the Black–Scholes one, shows that the expectation value of the stochastic option price satisfies the "macroscopic" Black–Scholes equation. In section 3 a master equation for the mesoscopic fair option value is constructed, and a stochastic simulation algorithm is applied to the valuation of European and American options. In the last section perspectives and further development of the approach are indicated.

2 Black–Scholes Equation

The standard textbook derivation of the Black–Scholes equation assumes a geometric Brownian motion for the underlying asset which is described by the stochastic differential equation

$$ds = \mu s\, dt + \sigma s\, dW \ . \tag{1}$$

Here s is the asset value, μs and $\sigma^2 s^2$ denote drift and variance of the random walk. Since dW is the increment of a Wiener process the resulting trajectory will be continuous. For more realistic models instantaneous jumps of the underlying, described by additional stochastic terms based on Poisson processes [21,22], may be included in (1). These so called jump diffusion models, are used e.g. to incorporate the effect of information about stocks which arrives at random times [6].

2.1 Black–Scholes equation from a piecewise deterministic process (PDP)

In contrast to the standard derivation of the Black–Scholes equation based on (1), this section demonstrates how the derivation can be paralleled by modelling the underlying in terms of a piecewise deterministic process [23]. This is possible since a diffusion process can be represented as the continuous limit of an appropriate jump process. This approach, despite describing the same dynamics as the standard approach, has the advantage of clearly demonstrating that it is quite obvious to interpret the option value as a stochastic variable and thus to interpret the Black–Scholes equation as the macroscopic expectation value of a mesoscopic stochastic option value. In addition the formulation in terms of PDPs simplifies the inclusion of additional jump processes since one has to deal only with one type of stochastic process. Another important point is that for such processes one can easily proceed to a master equation formulation, for which powerful numerical algorithms exist, as will be shown

later on.
Consider now the stochastic differential equation

$$ds = s\mu\, dt + \delta s\, dN^{+} - \delta s\, dN^{-}\ , \tag{2}$$

with $\langle dN^{\pm}\rangle = s^2\sigma^2/(2\delta s^2)\, dt$. The Poisson increments $dN^{\pm}$ are either 0, in which case the deterministic evolution takes place, or 1, which describes an instantaneous jump of size $\pm\delta s$. Stochastic process of this general form are thus also known as piecewise deterministic processes [24].
In the limit $\delta s \to 0$ this stochastic differential equation corresponds to a geometric Brownian motion with drift $s\mu$ and variance $s^2\sigma^2$ [25]. Hence in this limit the above equation generates the same dynamics as Eq. (1), but is formulated as a PDP.
Using the above stochastic differential Eq. (2) instead of Eq. (1) as starting point for the classical Black–Scholes derivation [6] one obtains in the limit $\delta s \to 0$ for the expectation value of the stochastic option price v

$$\langle\frac{\partial v(s,t)}{\partial t}\rangle = -\frac{s^2\sigma^2}{2}\langle\frac{\partial^2 v(s,t)}{\partial s^2}\rangle - rs\langle\frac{\partial v(s,t)}{\partial s}\rangle + r\langle v(s,t)\rangle\ , \tag{3}$$

where r denotes the riskfree interest rate. If the expectation value $V(S,t) = \langle v(s,t)\rangle$ is identified with the macroscopic option price then one obtains the Black–Scholes equation for a European option on a non dividend paying asset. Hence the traditional Black-Scholes equation can be interpreted as the expectation value of a stochastic process. Additional not infinitesimal jump processes can now easily be included, but this of course will change the hedging strategy [21,22] and the expectation value of the option price will then follow equations similar to the Black–Scholes equation.

2.2 Macroscopic Black–Scholes equation

The above derivation assumed a non-dividend paying stock, this assumption can be easily omitted to obtain [6,26]

$$\frac{\partial V(S,t)}{\partial t} = -\frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2} - (r-q)\, S\frac{\partial V(S,t)}{\partial S} + r\, V(S,t)\ . \tag{4}$$

Here σ and r again denote the volatility and the interest rate and the continuous dividend is given by q. In order to numerically solve this partial differential equation the relevant variables are made dimensionless and the time direction is reversed, i.e. we have,

$$\tilde{S} = \ln(S/E),\ \ \tilde{V}(\tilde{S},\tau) = V(S,t)/E,\ \ \tau = \frac{\sigma^2}{2}(T-t). \tag{5}$$

In this way one obtains

$$\frac{\partial\,\tilde{V}(\tilde{S},\tau)}{\partial\tau} = \frac{\partial^2\,\tilde{V}(\tilde{S},\tau)}{\partial\tilde{S}^2} + (k_q - 1)\frac{\partial\tilde{V}(\tilde{S},\tau)}{\partial\tilde{S}} - k_r\,\tilde{V}(\tilde{S},\tau)\,, \tag{6}$$

where $k_q = 2\,(r-q)/\sigma^2$ and $k_r = 2\,r/\sigma^2$ have been introduced. If the parameters of the Black–Scholes equation (4) are assumed to be constant in time, then the analytic solution to the above equation is available, see e.g. [6].

3 Master Equation formulation of the Black-Scholes Equation

As seen in section 2.1 the Black–Scholes equation can be interpreted as a deterministic equation which governs the expectation value of a stochastic option price. In this section we will describe how the stochastic dynamics of the option price can be formulated in terms of a master equation. For the sake of simplicity we will consider the Black-Scholes equation for a European call in the dimensionless form of Eq. (6). The generalisation to a put is straightforward and the application to American options is discussed in section 3.6.

3.1 General Theory

A general approach in statistical physics is to derive macroscopic equations from a known microscopic dynamics. Here the opposite approach is followed: given a partial differential equation for a macroscopic observable we construct a mesoscopic stochastic process such that the expectation value of the stochastic variable is governed by the original partial differential equation. In this section the general theory needed for the construction of a stochastic process underlying a partial differential equation is presented, for a review see [27].
In order to define the underlying stochastic process the state space of the system has to be given. To this end the space variable x is discretized such that θ_λ denotes the state of the system at the discrete position x_λ, $\lambda = 1,\ldots,n$. Hence the state space Γ of the system is given by $\Gamma = \{\theta \mid \theta \in \mathbb{R}^n\}$. The stochastic process is defined by the joint probability distribution $P = P(\{\theta_\lambda\},t)$ giving the probability of finding the values $\{\theta_\lambda\}$ at time t. The time development of the probability density P is given by a master equation of the general form

$$\frac{\partial}{\partial t}P(\theta,t) = \mathcal{A}P(\theta). \tag{7}$$

The operator $\mathcal{A}$ is defined in terms of diffeomorphic maps b acting on the state space $b : \Gamma \to \Gamma$, and corresponding operators $\mathcal{B}_b$ acting on functions

$F : \Gamma \to \mathbb{R}$ in the following way

$$\mathcal{B}_b F(\theta) = F(b^{-1}(\theta)). \tag{8}$$

The most general operator $\mathcal{A}$ needed in this paper is of the form

$$\mathcal{A} = c \sum_{\mu} [\mathrm{Det}(b_\mu)]^{-1} \mathcal{B}_{b_\mu} - \mathbb{I}, \tag{9}$$

where $\mathrm{Det}(b_\mu)$ denotes the determinant of the Jacobian matrix of the map b_μ and $\mathbb{I}$ denotes the identity. In order to generate a valid stochastic process the operator $\mathcal{A}$ has to fulfil (i) all transition probabilities w are positive and (ii) $\langle \mathcal{A} F \rangle = 0$ in order to preserve the normalisation of the probability density.
With these definitions the macroscopic equation of motion for a general observable F can now easily be computed, one obtains

$$\frac{\partial}{\partial t} \langle F \rangle = c \sum_{\mu} \langle F(b_\mu(\theta)) - F(\theta) \rangle. \tag{10}$$

The above equation enables the computation of the macroscopic expectation value of a general observable F given an arbitrary multivariate Markov process defined by the time evolution operator $\mathcal{A}$.
For a master equation formulation of the stochastic process one needs the transition probability

$$w(\theta, \tilde{\theta}) = c \sum_{\mu} \delta(\theta, b_\mu(\tilde{\theta})). \tag{11}$$

Hence the canonical form of the master equation is obtained

$$\frac{\partial}{\partial t} P(\theta, t) = \int D\tilde{\theta} \left\{ w(\theta, \tilde{\theta}) P(\tilde{\theta}, t) - w(\tilde{\theta}, \theta) P(\theta, t) \right\}. \tag{12}$$

Now the approach sketched above is applied to the Black-Scholes equation. Hence the interesting price range of the underlying $x = \tilde{S}$ is divided in n discrete points x_λ, $\lambda = 1, \ldots, n$ with distances $\Delta_\lambda = x_\lambda - x_{\lambda-1}$, $\lambda = 2, \ldots, n$.

3.2 Stochastic process with uniform discretization

The easiest possibility to construct the stochastic process is to use a uniform discretization of the state space $\Delta_\lambda = \delta l$. The Black-Scholes equation (6) consists of a diffusive, a convective and a part corresponding to a chemical reaction. The stochastic process underlying the Black-Scholes equation will also consist of three parts corresponding to these terms in the partial differential equation. The total time evolution operator $\mathcal{A}$ is thus defined by

$$\mathcal{A}_{\mathrm{BS}} = \mathcal{A}_{\mathrm{diff}} + \mathcal{A}_{\mathrm{conv}} + \mathcal{A}_{\mathrm{chem}}. \tag{13}$$

The above time evolution operators are defined through their corresponding maps, see Eq. (9). The map

$$a_\mu^\pm = \begin{cases} \theta_\mu & \to \theta_\mu - \alpha\,\theta_\mu \\ \theta_{\mu\pm1} & \to \theta_{\mu\pm1} + \alpha\,\theta_\mu \end{cases}, \tag{14}$$

describes the diffusive part, while the maps b_μ and c_μ

$$b_\mu = \begin{cases} \theta_\mu & \to \theta_\mu - \alpha\,\theta_\mu \\ \theta_{\mu-1} & \to \theta_{\mu-1} + \alpha\,\theta_\mu \end{cases}, \qquad c_\mu = \begin{cases} \theta_\mu \to \theta_\mu - \alpha\,\theta_\mu \end{cases}, \tag{15}$$

correspond to the convective part and to the chemical reaction term. Hence the time evolution operators for the three parts of the stochastic process are given by

$$\mathcal{A}_{\text{diff}} = \frac{1}{\alpha\delta l^2} \sum_\mu \frac{1}{1-\alpha}\{\mathcal{A}_\mu^+ + \mathcal{A}_\mu^-\} - 2\,\mathbb{I}, \tag{16}$$

$$\mathcal{A}_{\text{conv}} = \frac{k-1}{\alpha\,\delta l} \sum_\mu \frac{1}{1-\alpha}\mathcal{B}_\mu - \mathbb{I}, \tag{17}$$

$$\mathcal{A}_{\text{chem}} = \frac{k}{\alpha} \sum_\mu \frac{1}{1-\alpha}\mathcal{C}_\mu - \mathbb{I}. \tag{18}$$

Here the operators $\mathcal{A}_\mu^\pm, \mathcal{B}_\mu, \mathcal{C}_\mu$ are defined according to the general definition (8) through their maps $a_\mu^\pm, b_\mu, c_\mu$. In order to prove that the expectation value of the stochastic process whose generator is given by (13) really solves the Black-Scholes equation one uses Eq. (10) and the projection operator $F_\lambda(\theta) = \theta_\lambda$. One then immediately obtains

$$\frac{\partial}{\partial t}\langle\theta_\lambda\rangle = \frac{\langle\theta_{\lambda+1}\rangle + \langle\theta_{\lambda-1}\rangle - 2\langle\theta_\lambda\rangle}{\delta l^2} + (k-1)\frac{\langle\theta_{\lambda+1}\rangle - \langle\theta_\lambda\rangle}{\delta l} - k\langle\theta_\lambda\rangle, \tag{19}$$

which in the continuum limit $\delta l \to 0$ converges towards the dimensionless Black-Scholes equation (6).
With these definitions the transition probability $w(\theta, \tilde{\theta})$ becomes

$$\begin{aligned} w(\theta,\tilde{\theta}) = {} & \frac{1}{\alpha\delta l^2} \sum_\mu \delta(\theta, a_\mu^+(\tilde{\theta})) + \delta(\theta, a_\mu^-(\tilde{\theta})) + \frac{k-1}{\alpha\delta l} \sum_\mu \delta(\theta, b(\tilde{\theta})) \\ & + \frac{k}{\alpha} \sum_\mu \delta(\theta, c(\tilde{\theta})). \end{aligned} \tag{20}$$

Summarising the Black-Scholes Eq. (6) can be numerically solved by simulating the stochastic process

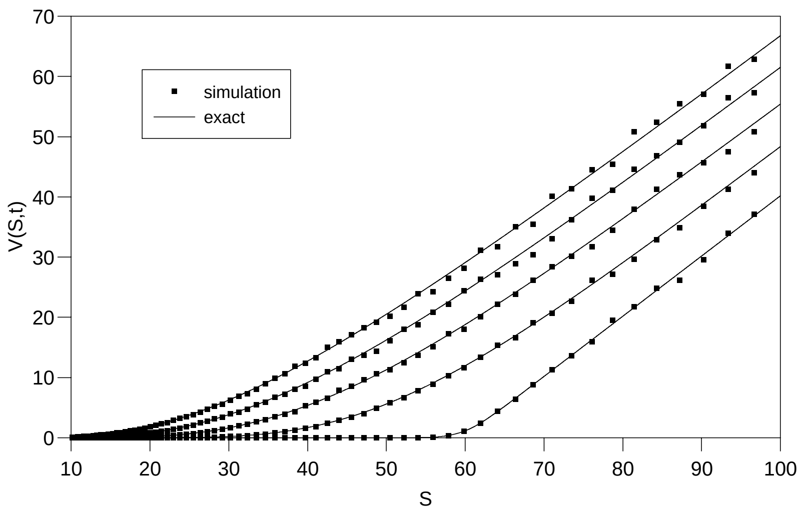

Fig. 1. Time evolution of the analytic solution (continuous line) and results of a direct stochastic simulation (squares) according to Eq. (13) of the Black-Scholes equation for an European call with Exercise price 60, $\sigma = 0.2$, $r = 0.06$ for five different points in time. The solution of the master equation was estimated by averaging 10 realization with $\alpha = 0.005$ at times $T = 10, 7.5, 5, 2.5, 0.05$.

$$\frac{\partial}{\partial t}P(\theta,t) = \mathcal{A}_{\rm BS}P(\theta) = \int D\tilde{\theta}\left\{w(\theta,\tilde{\theta})P(\tilde{\theta},t) - w(\tilde{\theta},\theta)P(\theta,t)\right\}. \tag{21}$$

This is usually done with the help of the following algorithm:

- initialise θ_λ at $t = 0$
- while $t < t_{\rm end}$
  - · compute random time step τ until the next jump occurs from an exponential distribution with mean value $\langle\tau\rangle = 1/w_{\rm tot}$
  - · apply one of the transitions $a_\mu^\pm, b_\mu, c_\mu$ selected according to their relative probability

By repeating the above algorithm different realizations of θ are generated and thus $\langle\theta_\lambda\rangle$ can be estimated from a sample of realizations.
This stochastic process has some interesting features. The parameter α can be used to control the size of the fluctuations. In earlier applications of the general theory presented above to balance equations of non-equilibrium statistical mechanics, the parameter α could be interpreted as the temperature of the physical system [27,28]. The larger the parameter α the larger are the fluctuations, hence it is intuitively clear that, e.g. in a thermodynamic setting α plays essentially the role of the temperature.
From a numerical point of view the total transition probability $w_{\rm tot}$ is constant. Thus the error made by taking a constant time step $\tau = 1/w_{\rm tot}$ in the numerical simulation vanishes as $O(\delta t)$. This significantly reduces the number of random variables needed. In addition the transition probabilities do not depend on the current state of the system. This makes the random selection of a transition very efficient, otherwise algorithms as discussed in [29] have to

be used.
Fig. 1 shows a typical result of a simulation averaged over 10 realizations of the stochastic process with $\alpha = 0.005$. Naturally the simulation results fluctuate around the analytic solution, but these fluctuations are very small if one takes into account that only 10 realizations are averaged. The reason for this of course is the parameter α. The smaller α, the smaller are the fluctuations of the process and the smaller is the number of realizations needed.
One drawback of this stochastic process is, that it is restricted to a uniform discretisation of the dimensionless variable $\tilde{S}$ from Eq. (6). As can be seen in Fig. 1 after transforming back to the original variables V and S this leads to an exponential distribution of the discrete values of the underlying, see Eqs. (5). Hence most of the points are at the left border of the integration region. Since the total transition rate performs as $1/\delta l^3$ the length of the time step during the simulation and thus the overall computing time depends strongly on the discretisation of the underlying. A numerically more efficient algorithm thus has to use a uniform discretization of the underlying S which hence requires much less points. This results in a non-uniform discretization of the dimensionless variable $\tilde{S}$ in Eq. (6).

3.3 Stochastic process with non-uniform discretization

To enable a non-uniform discretization of the Black-Scholes equation (6) one uses an arbitrary distance Δ_λ of two neighbouring points. The time evolution operator $\mathcal{A}_{\text{chem}}$ from the previous section does not depend on the discretization and is thus not altered. But for the diffusive and convective part of the Black-Scholes equation new stochastic processes $\mathcal{A}_{\text{diff}}^{\text{n.u.}}$ and $\mathcal{A}_{\text{conv}}^{\text{n.u.}}$ taking into account the non-uniform (n.u.) discretisation have to be defined. With the definitions

$$r_\mu^+ = \begin{cases} \theta_\mu & \to \theta_\mu - \alpha_1 \frac{\theta_\mu}{\Delta_{\mu-1}\Delta_\mu} \\ \theta_{\mu+1} & \to \theta_{\mu+1} + \alpha_1 \frac{2\theta_\mu \Delta_{\mu+1}}{(\Delta_{\mu+1}+\Delta_\mu)\Delta_\mu \Delta_{\mu+1}} \end{cases}, \tag{22}$$

$$r_\mu^- = \begin{cases} \theta_\mu & \to \theta_\mu - \alpha_1 \frac{\theta_\mu}{\Delta_{\mu-1}\Delta_\mu} \\ \theta_{\mu-1} & \to \theta_{\mu-1} + \alpha_1 \frac{2\theta_\mu \Delta_{\mu-2}}{(\Delta_{\mu-1}+\Delta_{\mu-2})\Delta_{\mu-2} \Delta_{\mu-1}} \end{cases}, \tag{23}$$

for the diffusive and

$$s_\mu = \begin{cases} \theta_\mu & \to \theta_\mu - \alpha_2 \frac{\theta_\mu}{\Delta_\mu} \\ \theta_{\mu-1} & \to \theta_{\mu-1} + \alpha_2 \frac{\theta_\mu}{\Delta_{\mu-1}} \end{cases}, \tag{24}$$

for the convective part one obtains the two new time evolution operators

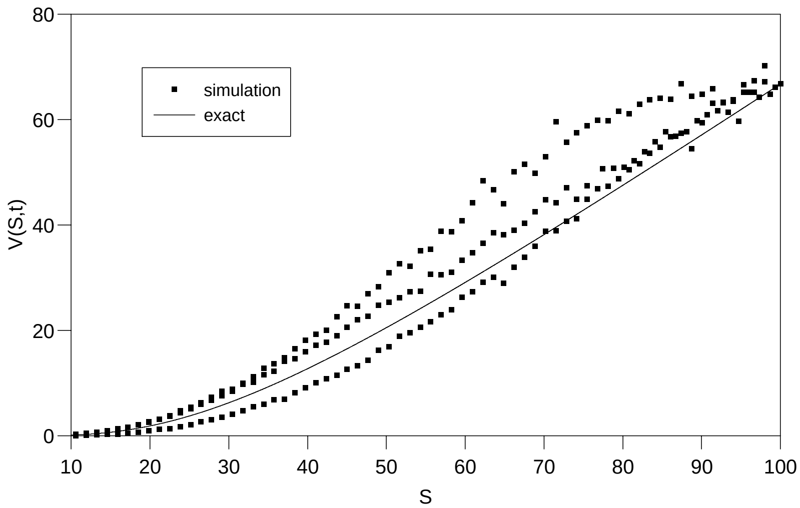

Fig. 2. Three exemplary realizations of the stochastic process from Eq. (33) (squares) and the corresponding exact solution (continuous line) of the Black–Scholes equation with Exercise price 60, $\sigma = 0.2$, $r = 0.06$ and $T = 10$.

$$\mathcal{A}^{\text{n.u.}}_{\text{diff}} = \frac{1}{\alpha_1} \sum_{\mu} \frac{1}{\text{Det}(r^+_\mu)} \mathcal{R}^+_\mu + \frac{1}{\text{Det}(r^-_\mu)} \mathcal{R}^-_\mu - 2\,\mathbb{I}, \tag{25}$$

$$\mathcal{A}^{\text{n.u.}}_{\text{conv}} = \frac{k-1}{\alpha_2} \sum_{\mu} \frac{1}{\text{Det}(s_\mu)} \mathcal{S}_\mu - \mathbb{I}. \tag{26}$$

The total time evolution operator $\mathcal{A}^{\text{n.u.}}_{\text{BS}}$ is now given by

$$\mathcal{A}^{\text{n.u.}}_{\text{BS}} = \mathcal{A}^{\text{n.u.}}_{\text{diff}} + \mathcal{A}^{\text{n.u.}}_{\text{conv}} + \mathcal{A}_{\text{chem}}. \tag{27}$$

In order to prove that the expectation value $\langle \theta_\lambda \rangle$ of the stochastic process above really solves the Black-Scholes equation one again introduces the projection operator F_λ and makes use of the general theorem (10) to obtain

$$\begin{aligned} \frac{\partial}{\partial t} \langle \theta_\lambda \rangle = & \frac{\frac{2\Delta_{\lambda-1}}{\Delta_\lambda + \Delta_{\lambda-1}} \langle \theta_{\lambda+1} \rangle + \frac{2\Delta_\lambda}{\Delta_\lambda + \Delta_{\lambda-1}} \langle \theta_{\lambda-1} \rangle - 2\langle \theta_\lambda \rangle}{\Delta_{\lambda-1}\Delta_\lambda} \\ & + (k-1) \frac{\langle \theta_{\lambda+1} \rangle - \langle \theta_\lambda \rangle}{\Delta_{\lambda+1}} - k\,\langle \theta_\lambda \rangle. \end{aligned} \tag{28}$$

In the continuum limit the expectation value $\langle \theta_\lambda \rangle$ of this stochastic process thus again solves the Black-Scholes equation, but now on a non-uniform grid. The transition probability $w(\theta, \tilde{\theta})$ becomes

$$\begin{aligned} w(\theta, \tilde{\theta}) = & \frac{1}{\alpha_1} \sum_\mu \delta(\theta, r^+_\mu(\tilde{\theta})) + \delta(\theta, r^-_\mu(\tilde{\theta})) + \frac{k-1}{\alpha_2} \sum_\mu \delta(\theta, s(\tilde{\theta})) \\ & + \frac{k}{\alpha} \sum_\mu \delta(\theta, c(\tilde{\theta})). \end{aligned} \tag{29}$$

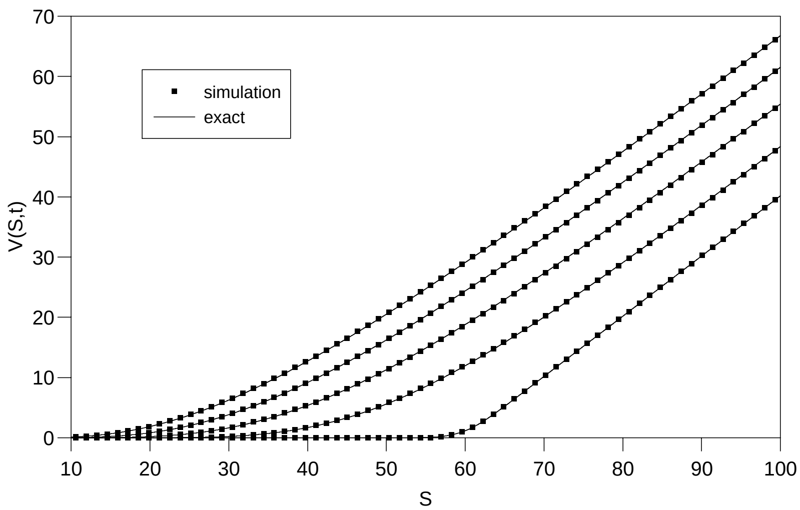

Fig. 3. Time evolution of the analytic solution (line) and results of a direct stochastic simulation (squares) according to Eq. (33) averaged over 200 realizations of the Black-Scholes equation for an European call with Exercise price 60, $\sigma = 0.2$, $r = 0.06$ at the time points $T = 10, 7.5, 5, 2.5, 0.05$.

The parameters α_i are chosen such that the relative size of the transitions is smaller than 1, hence they fulfil $\alpha < 1$, $\alpha_2/\Delta_\mu < 1$ and $2\alpha_1/\Delta_\mu^2 < 1$ for every possible transition μ. The smallest α_i, which is α_1, determines the scaling of the total transition probability

$$w = \frac{2n}{\alpha_1} + (k-1)\frac{n}{\alpha_2} + \frac{k}{\alpha}. \tag{30}$$

Since $\alpha_1 \sim O(\Delta_\mu^2)$ and the total transition probability scales as $O(n/\alpha_1)$ the numerical effort to simulate the stochastic process scales according to $O(n/\Delta_\mu^2)$. This is the same dependence on the discretization as in the previous section with a stochastic process on a uniform grid. But since now the grid can be chosen appropriate one needs much less grid points and the algorithm is much faster.

3.4 Fast stochastic process with non-uniform discretization

The use of a non-uniform grid reduces the number of grid points to be used in a numerical simulation. But there is still a $O(n/\Delta_\mu^2) \approx O(1/\Delta_\mu^3)$ dependence of the total transition rate which makes the algorithm slow. To get rid of this dependence the stochastic process corresponding to the diffusive part is again improved. In analogy to equilibrium Monte Carlo simulation of lattice systems [30,31], where one generates a new configuration of the whole lattice in a single

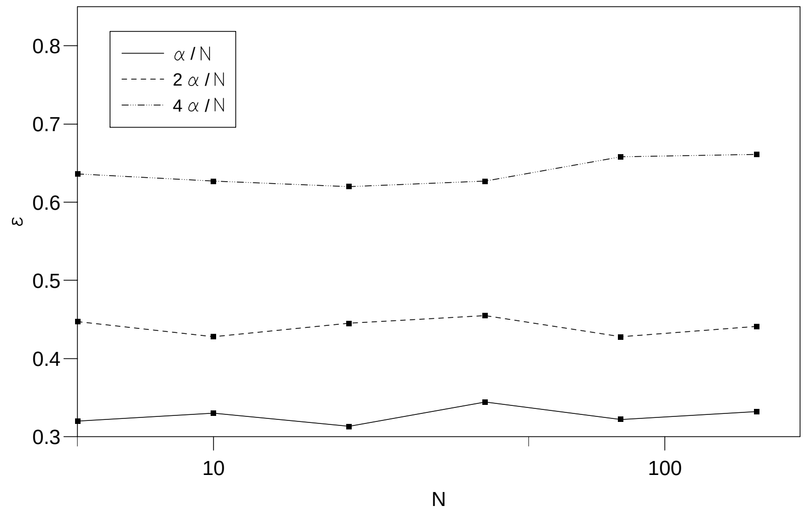

Fig. 4. Numerical root mean square error ϵ of the solution of the Black–Scholes equation for an European call with Exercise price 60, $\sigma = 0.2$, $r = 0.06$ according to the master equation (33) for three different ratios α/N. The parameter values for the first simulation (α/N) are $\alpha_1/N = 10^{-7}$, $\alpha_2/N = 10^{-5}$ and $\alpha_3/N = 10^{-4}$.

sweep, one defines

$$\mathcal{A}_{\text{diff}}^{\text{sweep}} = \frac{1}{\alpha_1}\left[\frac{1}{\text{Det}(t^+)}\mathcal{T}^+ + \frac{1}{\text{Det}(t^-)}\mathcal{T}^- - 2\,\mathbb{I}\right], \tag{31}$$

with the map $t^{\pm}$ given by

$$t^{\pm} = \sum_{\mu} r_{\mu}^{\pm}. \tag{32}$$

This process thus updates every θ_λ at once. The total time evolution operator is now given by

$$\mathcal{A}_{\text{BS}}^{\text{sweep}} = \mathcal{A}_{\text{diff}}^{\text{sweep}} + \mathcal{A}_{\text{conv}}^{\text{n.u.}} + \mathcal{A}_{\text{chem}}. \tag{33}$$

For the transition probability one obtains

$$\begin{aligned} w(\theta, \tilde{\theta}) = {} & \frac{1}{\alpha_1}\,\delta(\theta, t^+(\tilde{\theta})) + \frac{1}{\alpha_1}\,\delta(\theta, t^-(\tilde{\theta})) \\ & + \frac{k-1}{\alpha_2}\sum_{\mu}\delta(\theta, s(\tilde{\theta})) + \frac{k}{\alpha}\sum_{\mu}\delta(\theta, c(\tilde{\theta}))\ . \end{aligned} \tag{34}$$

Since α_1 again scales as $O(1/\Delta_\mu^2)$ the total transition probability now also scales according to $O(1/\Delta_\mu^2)$. This of course makes the algorithm significantly faster.

Fig. 2 shows three exemplary solutions of the stochastic process defined in (33) which corresponds to a non-uniform discretisation of the dimensionless underlying $\tilde{S}$. In addition to the expected random fluctuations around the exact solution, single realizations also show systematic deviations from the exact solution for a wide range of values of the underlying asset. These systematic and random fluctuations can be traced back to the two different types of stochastic processes entering Eq. (33). The generators $\mathcal{A}^{\text{n.u.}}_{\text{conv}}$ and $\mathcal{A}_{\text{chem}}$ describe single jump processes, where the jump process changes only one option value in asset time space, this of course causes the random fluctuations. In contrast the systematic deviations stem from $\mathcal{A}^{\text{sweep}}_{\text{diff}}$ which describes a jump process changing all option values at once.
The numerical estimation of the option price from a sample of realizations of the stochastic process is of course not affected by this behaviour of single realizations. By averaging about several realizations the estimated option price converges against the analytic solution, this is shown in Fig. 3.
Concerning the numerical root mean square error error

$$\epsilon = \sqrt{\sum_k \left[V(S_k,t) - \hat{V}(S_k,t)\right]^2}, \tag{35}$$

of the Monte Carlo simulation of the Black–Scholes equations one has to distinguish two types of parameters entering the numerical algorithm derived in the previous section: the number of samples N used to estimate the solution $\hat{V}$ and the parameters α_i which control the size of the fluctuations. Figure 4 shows the root mean squared error ϵ of a Monte Carlo simulation for three different values of the ratio α_i/N. Different simulations of the same ratio were obtained e.g. by decreasing N and all three parameters α_i by the same factor. This figure clearly shows that the error only depends on this ratio α_i/N. In addition it can be seen that the error scales with the square root of this ratio. This of course is a dependence which is expected for a Monte Carlo simulation.
In order to assess the numerical performance of the proposed Monte Carlo method it is compared with the standard Monte Carlo approach based on the simulation of the dynamics of the underlying asset. The numerical error in a Monte Carlo solution has two sources. The systematic part of the error stems from the finite discretization of the stochastic differential equation in the case of the simulation of the geometric Brownian motion and from the finite discretization of the state space when one solves the master equation. In addition there is a random error from the averaging over different solutions which decays in both cases as $N^{-0.5}$ (if α is fixed in the master equation formulation). Figure 5 shows the time required to achieve a given precision for the two methods in a parameter range where the systematic part of the numerical errors are comparable. As can be easily seen the solution of the

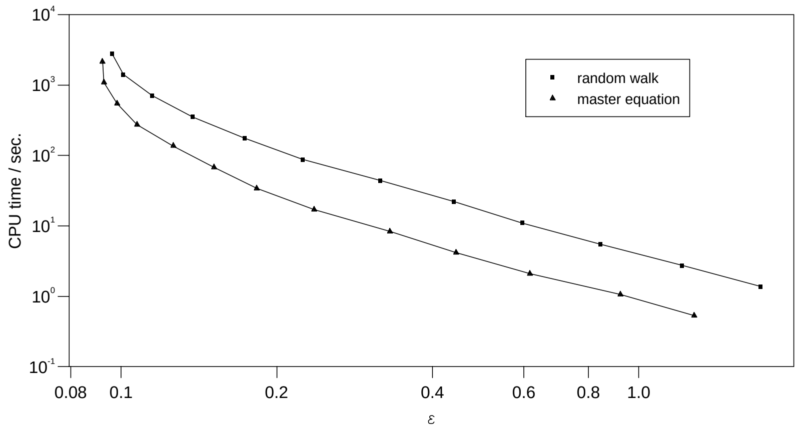

Fig. 5. Comparison of the computation time in seconds needed for the pricing of an option by simulation of the stochastic dynamics of the underlying asset and by a master equation formulation of the Black–Scholes equation. Shown is the computation time vs. the numerical root mean square error ϵ.

master equation is significantly faster than the standard approach.

3.6 American options

Up to now only European options have been considered. How can the concept of mesoscopic Monte Carlo simulations be generalised for American options? A straightforward generalisation of the algorithm simulating the time evolution of the underlying asset is quite involved, since the early exercise condition has to be fulfilled. Whenever the option price falls below the payoff function the option is exercised and the option value is thus given by the payoff. Hence in order to assure that the early exercise condition is not violated sometime in the future a straightforward generalisation of this Monte Carlo simulation would have to keep track of all these points in asset time space which would make the algorithm very ineffective. Of course, there are more advanced techniques to generalise the Monte Carlo approach to American options, see [10] for an overview.

The proposed mesoscopic approach based on a master equation whose expectation value solves the original Black–Scholes equation can be generalised to the valuation of American options. Since the time direction in the dimensionless Eq. (6) has been reversed and the payoff function is used as initial condition the generalisation to American options is straightforward. One just has to simulate one time step according to the master equation (33) and average about the number of samples used to estimate the option price. Wherever this option price is below the payoff function it is replaced with the payoff and the next time step is simulated. Hence one assures that the early exercise condition is fulfilled.

For this approach to work it is critical that first the samples are averaged

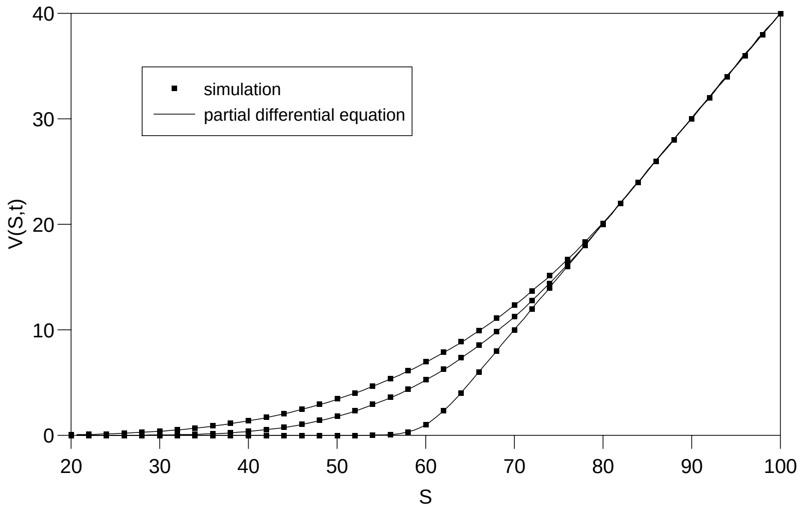

Fig. 6. Time evolution for an American call with Exercise price 60, interest rate 0.07 and dividend yield 0.10 according to the stochastic simulation (squares) and the solution of the corresponding partial differential equation (continuous line) for the time points $T = 10, 7.5, 5$.

and that the early exercise condition is applied thereafter. Applying the early exercise condition before averaging would introduce a bias towards higher option prices. Every time the stochastic realization is by chance below the payoff function the option value is increased. But since the option value is never decreased this way it is intuitively clear that one would obtain higher option prices. Fig. 6 shows the result of such a simulation for an American call with exercise price 60, interest rate 0.07, volatility 0.2 and a continuous dividend yield of 0.10.

4 Conclusions

In real markets the elegance of the perfect hedge of the Black–Scholes approach to option pricing is generally lost. Already a very simple model of the dynamics of the underlying stock in terms of piecewise deterministic processes shows that the option price itself may be regarded as a stochastic variable. However the dynamics of the expectation value of this stochastic option price is governed by the Black–Scholes equation. This consideration is the motivation for the master equation approach to option pricing suggested in this paper.

The essential point of this approach is that the Black–Scholes equation is interpreted as the macroscopic equation of an underlying mesoscopic stochastic process for the stochastic option price variable. By using PDPs one can then easily proceed to a master equation formulation of the option pricing problem. In contrast to this the usually used microscopic approach describes the dynamics of the underlying asset by a stochastic differential equation.

The master equation formulation of the option pricing theory offers several advantages over the standard approach. This formulation provides a general setting in which also other kinds of jump processes may be easily embedded, without altering the character of the equations. One may also include additional stochastic processes for the volatility and the interest rate and then arrive at stochastic volatility and interest rate models [32,33]. This of course would lead to different hedging strategies. One possibility to extend the proposed approach beyond the Black–Scholes equation, which is currently under investigation, is to use a hedging strategy proposed in [34,35]

In addition the master equation formulation also allows for the use of advanced simulation algorithms. It was shown that it is possible to construct a master equation whose transition probability is constant in time and does not depend on the current state of the system. As shown in section 3.1 this allows for fast numerical algorithms for the computation of the option price.

Using the standard Black–Scholes equation as an example we have demonstrated how to construct such numerically efficient stochastic processes underlying the partial differential equation. As can be seen from the derivation of the general theory in section 3.1, it is clear that this approach is not restricted to the standard Black–Scholes equation but can be applied to generalisations of the latter.

The master equation formulation of the Black–Scholes equation is numerically about a factor of two faster than the standard Monte Carlo approach. This is mainly due to the fact that the proposed mesoscopic approach does not simulate sample trajectories of the underlying asset which, after averaging, results in the option price for the initial asset value at a given time. In contrast to this the mesoscopic approach works in the whole asset time space and generates sample option prices in the whole state space during one realization and is hence much faster.

Another advantage of the master equation formulation of the Black–Scholes equation is that it allows for a straightforward generalisation to price American options in the same framework by just comparing the average option price with the payoff function.

Summarising, the proposed master equation approach is located in between the standard Monte Carlo approach for the simulation of the underlying asset and the finite difference solution of the partial differential equation. This approach tries to combine the numerical efficiency of the solution of the partial differential equation with the advantages of the Monte Carlo approach. The main advantage of the Monte Carlo approach is that it can easily be applied to price options using also advanced stochastic models for the underlying.

Concluding, the mesoscopic approach we have formulated has been exploited to implement a master equation approach to option pricing. Of course, it will be of great interest to investigate in further work if features of real markets correspond to the additional freedom to choose the size of the fluctuations contained in the proposed master equation formulation. This will eventually lead to a better understanding of option pricing in real markets.

References

[1] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[2] J. C. Hull. *Options, Futures, and other Derivatives.* Prentice Hall International, London, 1997.

[3] Bruno Dupire. *Monte Carlo: Methodologies and Applications for Pricing and Risk Management.* Risk Books, London, 1998.

[4] P. Boyle. Options: A Monte Carlo Approach. *Journal of Financial Economics*, 4:323–338, 1977.

[5] K. K. Sabelfeld. *Monte Carlo Methods in Boundary Value Problems.* Springer, 1991.

[6] P. Wilmott. *Derivatives, The Theory and Practice of Financial Engineering.* John Wiley & Sons, Chichester, 1998.

[7] F. A. Longstaff and E. S. Schwartz. Valuing american options by simulation: a simple least-squares approach. *Rev. Financ. Stud.*, 14(1):113–147, 2001.

[8] M. Broadie and P. Glasserman. Pricing American-style Securities using Simulation. Technical report, Columbia University, 1997.

[9] P. Bossaerts. Simulation Estimators of Optimal Early Exercise. Technical report, Carnegie Mellon University, 1989.

[10] P. Boyle, M. Broadie, and P. Glasserman. Monte Carlo Methods for Security Pricing. *Journal of Economic Dynamics & Control*, 21:1267–1321, 1997.

[11] H.-P. Breuer and F. Petruccione. Burgers's turbulence model as a stochastic dynamical system: Master equation and simulation. *Phys. Rev. E*, 47:1803–1814, 1993.

[12] H.-P. Breuer and F. Petruccione. Stochastic simulations of high-Reynolds-number turbulence in two dimensions. *Phys. Rev. E*, 50:2795–2801, 1994.

[13] P. Biechele, H.-P. Breuer, and F. Petruccione. Non-equilibrium Monte Carlo simulation of a decaying Navier-Stokes turbulence. *Phys. Lett. A*, 256:147–152, 1999.

[14] H.-P. Breuer and F. Petruccione. A Master equation formulation of fluctuating hydrodynamics. *Physica A*, 192:569–588, 1993.

[15] H.-P. Breuer and F. Petruccione. A master equation description of fluctuating hydrodynamics. *Physica A*, 192:569–588, 1993.

[16] R. Grauer and C. Marliani. Analytical and numerical approaches to structure functions in magnetohydrodynamic turbulence. *Physica Scripta*, T67:38–42, 1996.

[17] H.-P. Breuer, W. Huber, and F. Petruccione. Fast monte carlo algorithm for nonequilibrium systems. *Phys. Rev. E*, 53:4232–4235, 1996.

[18] H. P. Breuer, J. Honerkamp, and F. Petruccione. A stochastic approach to complex chemical reactions. *Chem. Phys. Lett.*, 190:199–201, 1992.

[19] H. P. Breuer, W. Huber, and F. Petruccione. Fluctuation effects on wave propagation in a reaction–diffusion process. *Physica D*, 73:259–273, 1994.

[20] N. G. van Kampen. *Stochastic Processes in Physics and Chemistry*. North-Holland, Amsterdam, 1992.

[21] R. C. Merton. Option Pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

[22] J. C. Cox. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166, 1976.

[23] M. H. A. Davis. *Markov Models and Optimization*. Chapman & Hall, London, 1993.

[24] H. P. Breuer and F. Petruccione. *The Theory of Open Quantum Systems*. Oxford University Press, Oxford, 2002.

[25] C. W. Gardiner. *Handbook of Stochastic Methods*. Springer Verlag, Berlin, 1990.

[26] P. Wilmott, S. Howison, and J. Dewynne. *The Mathematics of Finacial Derivaties, A Student Introduction*. Cambridge University Press, Cambridge, 1995.

[27] H.-P. Breuer and F. Petruccione. How to build master equations for complex systems. *Continuum Mech. Thermodyn.*, 7:439–473, 1995.

[28] H.-P. Breuer and F. Petruccione. Thermostochastics: Heath conduction and temperature fluctuations. *Physica A*, 209:83–95, 1994.

[29] M. Fricke and J. Schnakenberg. Monte-carlo simulation of an inhomogeneous reaction-diffusion system in the biophysics of receptor cells. *Z. Phys. B*, 83:277–284, 1991.

[30] K. Binder. *The Monte Carlo Method in Condensed Matter Physics*. Springer Verlag, Berlin, 1995.

[31] D. P. Landau and K. Binder. *A Guide to Monte Carlo Simulation in Statistical Physics*. Cambridge University Press, Cambridge, 2000.

[32] J. Hull and A. White. The Pricing of Options on Assets with Stochastic Volatilities. *The Journal of Finance*, 42(2):281–300, 1987.

[33] L. O. Scott. Option Pricing when the Variance Changes Randomly: Theory, Estimation, and an Application. *Journal of Financial and Quantitative Analysis*, 22(4):419–438, 1987.

[34] J. P. Bouchaud and M. Potters. *Theory of financial risks*. Cambridge University Press, Cambridge, 2000.

[35] J. P. Bouchaud, M. Potters, and D. Sestovic. Hedged Monte–Carlo: low variance derivative pricing with objective probabilities. *Physica A*, 289:517–525, 2001.